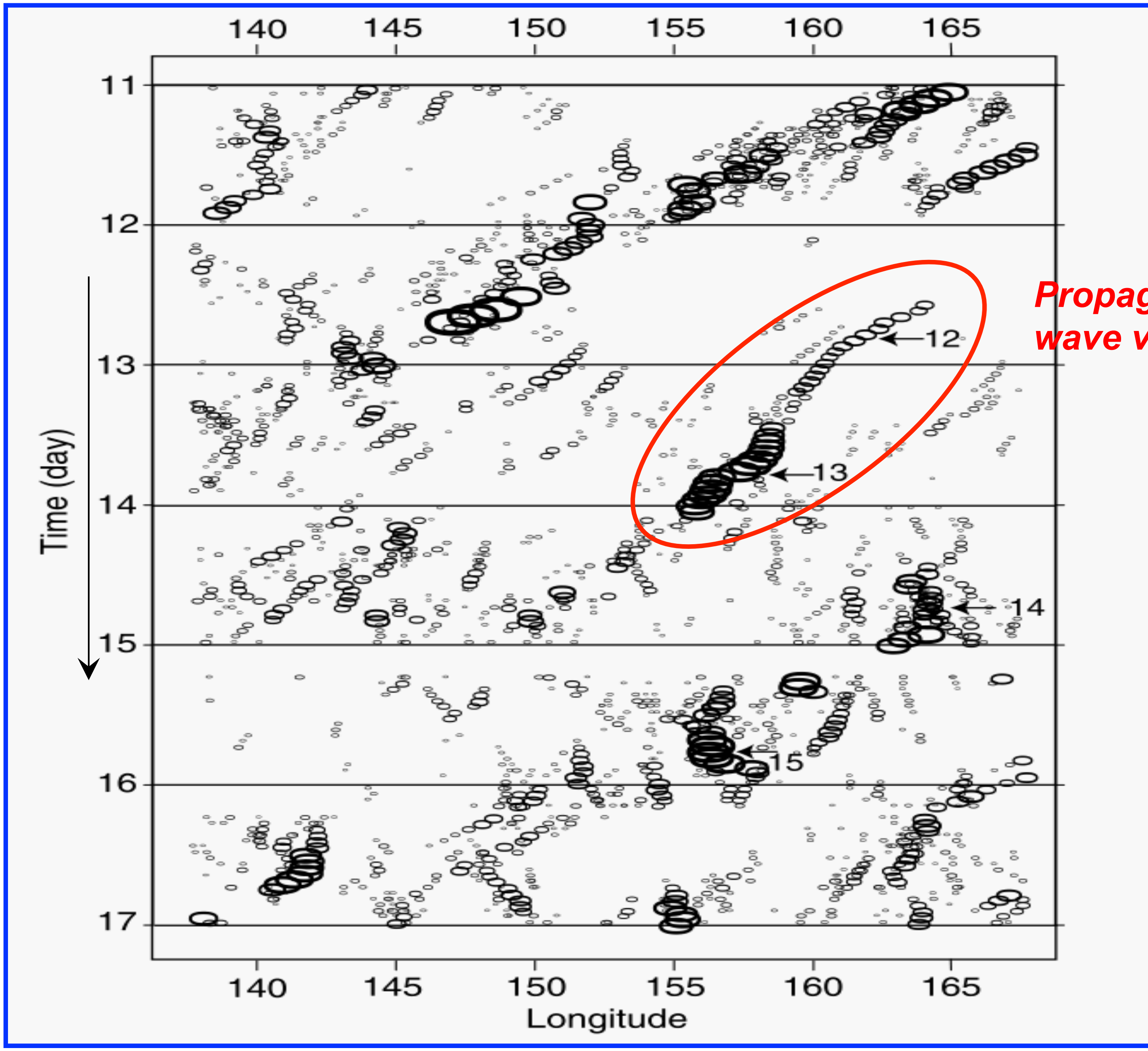
- Some examples of tropical waves
- Tropical planetary waves and the shallow water equations

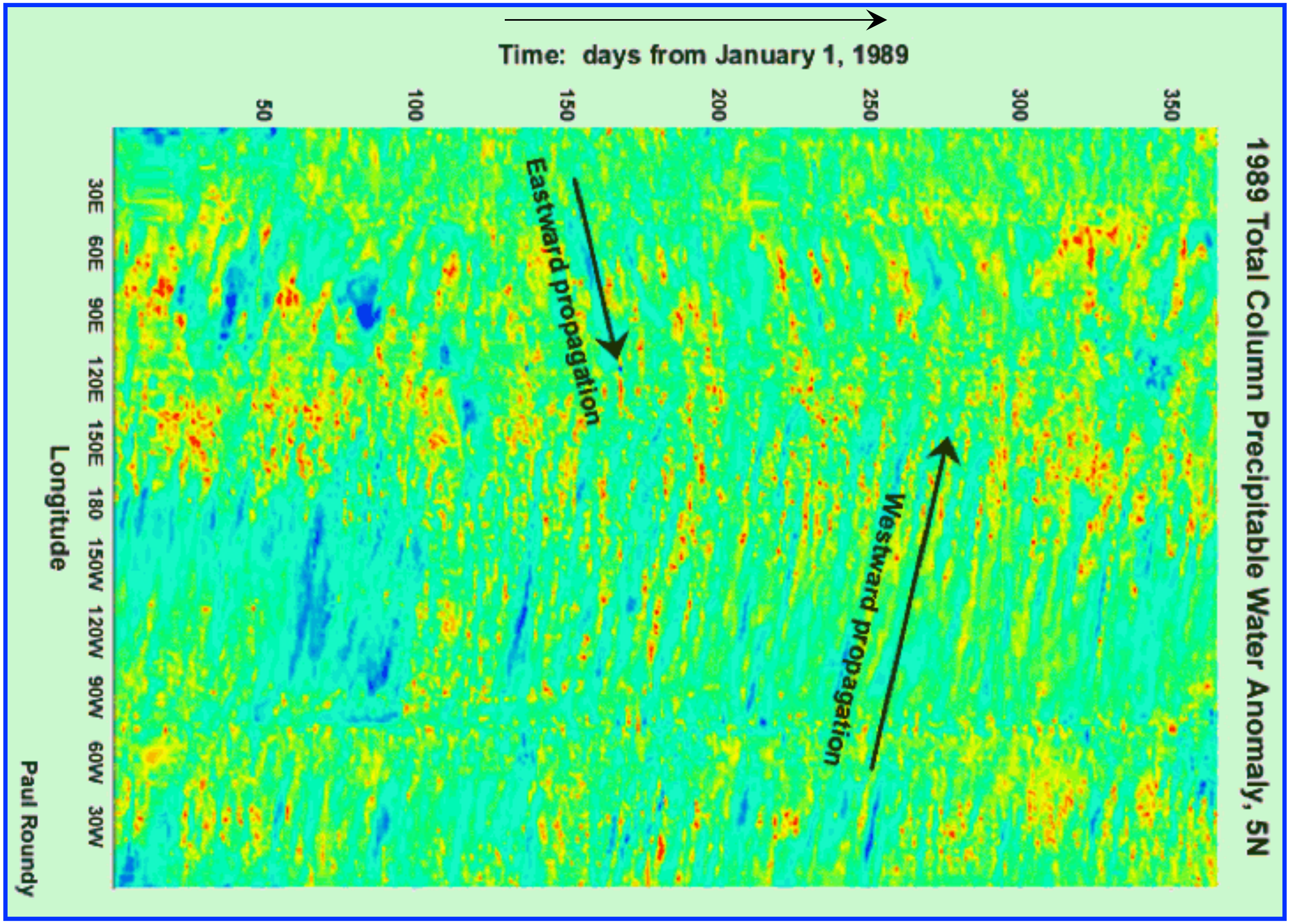
Goal: Develop the physical and mathematical foundations of tropical waves

Previously: MCS Hovmoller



From Chen et al (1996)

Precipitable Water Anomalies



Kelvin Wave

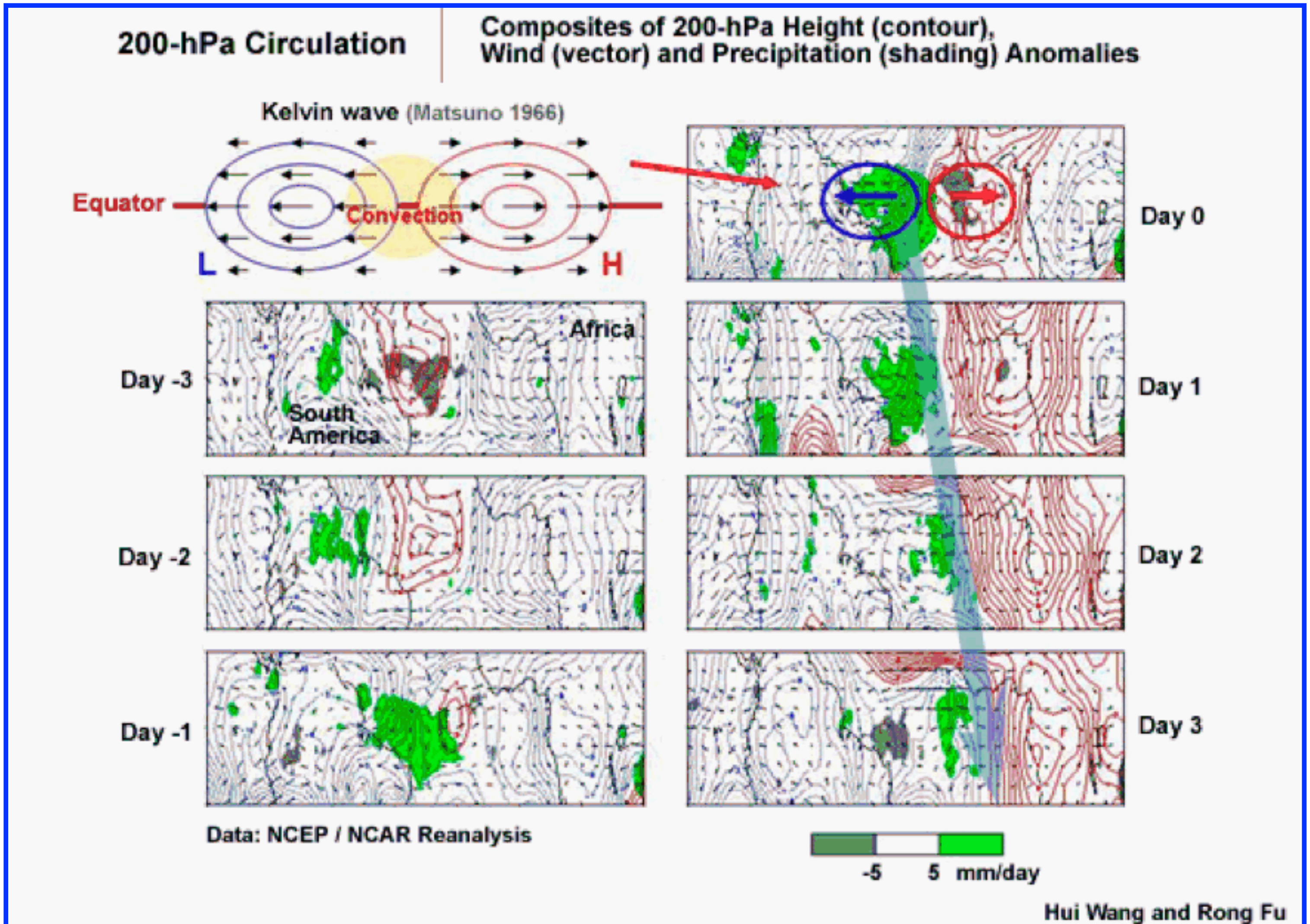


Fig. 5.15. Composite of 200 hPa height (contour), wind (vector) and precipitation (shading) anomalies. Top left shows the theoretical Kelvin wave derived by Matsuno (1966).[13] Light blue arrow shows the eastward movement of the wave.

Rossby Wave

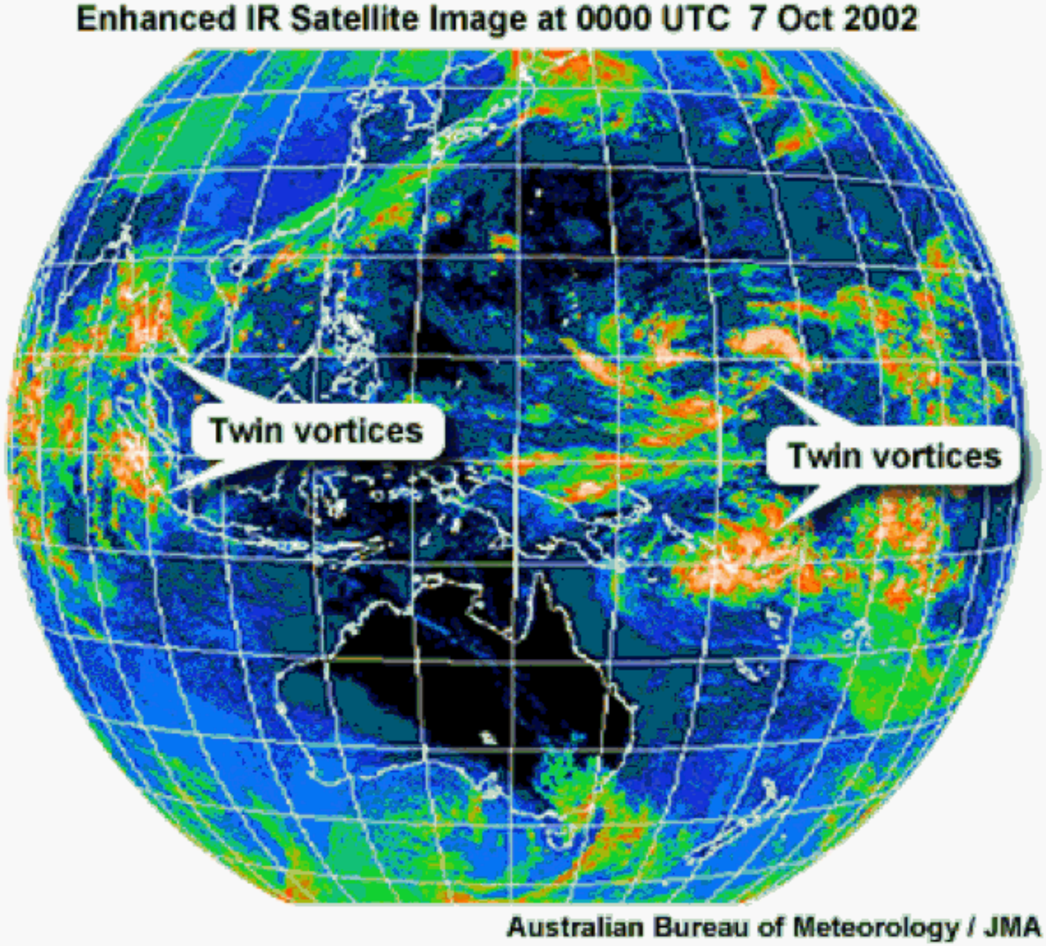


Fig. 5.17. Enhanced IR satellite image corresponding to the 850 hPa tropical wind analysis for 0000 UTC 7 October 2002, pictured in Fig. 5.16. Satellite image originally processed by the Bureau of Meteorology from the geostationary satellite GMS-5 operated by the Japan Meteorological Agency.

Dry adiabatic primitive equations

Recall from earlier lectures:

Vector momentum equation in rotating coordinates

$$\frac{d\vec{v}}{dt} = -2\vec{\Omega} \times \vec{v} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Continuity:

$$\nabla \cdot \vec{v} = 0$$

To derive mathematical expressions of the various types of large-scale tropical waves, we invoke a simplified form of these equations–the shallow water equations [SWE]. The approach here follows Matsuno 1966.

SWE: Principal Assumptions

- The atmosphere [or ocean] is approximated by two layers, both of which are horizontally and vertically homogeneous at rest.
- In the upper layer of the atmosphere, both pressure and density are horizontally invariant; in the lower layer, only density is constant.
- The fluid is hydrostatic.

SWE Derivation (I)

From the hydrostatic assumption:

$$\frac{\partial p}{\partial z} = -\rho g$$

Taking the horizontal gradient of the hydrostatic equation in either layer gives:

$$\nabla_H\left(\frac{\partial p}{\partial z}\right) = \nabla_H\left(-\rho g\right) = 0 = \frac{\partial}{\partial z}\left(\nabla_H p\right)$$

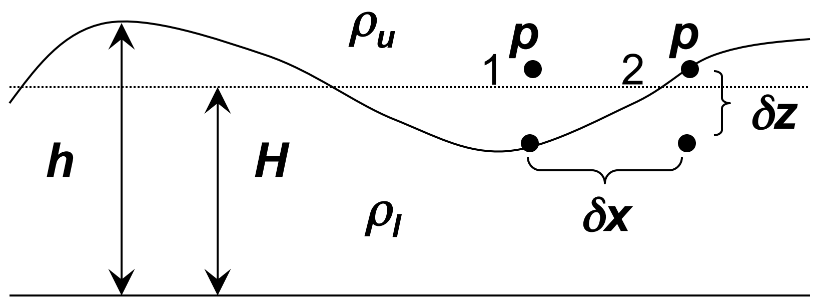
with the last equality following from the horizontal invariance of density in either layer. The last equality implies that the lower level horizontal pressure gradient is also invariant with height.

It is convenient to replace pressure by the depth of fluid, *h*, in the lower layer. That is:

$$\int_0^h \frac{\partial}{\partial z}\left(\nabla_H p\right)dz = \nabla_H \int_0^h \frac{\partial p}{\partial z}dz = -\nabla_H \int_0^h \rho g\,dz = -\rho g \nabla_H h$$

SWE Derivation (II)

Consider the set-up as illustrated schematically on the right. Here, we take *h(x,y,t)* = *H* + *η(x,y,t)*. We need to estimate horizontal variations in pressure in the lower layer.



For points 1 and 2 we have [considering, w/o loss of generality, x only]:

$$p_{l1} = p + \delta p_1 = p + \rho_u g \delta z$$

$$p_{l2} = p + \delta p_2 = p + \rho_l g \delta z$$

Thus:

$$\lim_{\delta x \to 0}\left[\frac{(p + \rho_l g \delta z) - (p + \rho_u g \delta z)}{\delta x}\right] = g(\rho_l - \rho_u)\lim_{\delta x \to 0}\frac{(\partial h / \partial x)\delta x}{\delta x} = g(\rho_l - \rho_u)\frac{\partial h}{\partial x}$$

The horizontal momentum equations in the lower layer become:

$$\frac{\partial u}{\partial t} + \vec{v}_H \cdot \nabla_H u - fv = -\frac{1}{\rho_l}\frac{\partial p_l}{\partial x} = -g'\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + \vec{v}_H \cdot \nabla_H v + fu = -\frac{1}{\rho_l}\frac{\partial p_l}{\partial y} = -g'\frac{\partial \eta}{\partial y}$$

Here, g′ is *reduced gravity*:

$$g' = g\frac{\rho_l - \rho_u}{\rho_l}$$

SWE Derivation (III)

From the continuity equation:

$$\frac{\partial w}{\partial z} = -\nabla_H \cdot \vec{v}_H$$

Integrating from the surface up to *h*, and noting that the horizontal velocity is independent of height [since the pressure gradient is height independent] yields:

$$\int_0^h \frac{\partial w}{\partial z} dz = w(h) - w(0) = -\int_0^h \nabla_H \cdot v_H \, dz = -h\nabla_H \cdot v_H$$

The vertical velocity vanishes at the surface. On the other hand, *w(h)* is the rate at which the interfacial height changes, i.e.,

$$w(h) = \left(\frac{\partial}{\partial t} + \vec{v}_H \cdot \nabla_H \right) h$$

Thus, using $h(x,y,t) = H + \eta(x,y,t)$, gives:

$$\frac{\partial \eta}{\partial t} = -\vec{v}_H \cdot \nabla_H \eta - (H + \eta) \nabla_H \cdot \vec{v}_H$$

Linearizing the SWE

We want to solve the SWE in perturbation form, i.e., the wave solutions are calculated with respect to a background state (*U*, *V*, and N):

$$\begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \begin{pmatrix} U + u' \\ V + v' \\ \mathrm{N} + \eta' \end{pmatrix}$$

For a motionless background state ($U=V=N=0$) and ignoring terms in quadratic in perturbation quantities gives:

$$\frac{\partial u'}{\partial t} = fv' - \frac{\partial \phi'}{\partial x}$$

$$\frac{\partial v'}{\partial t} = -fu' - \frac{\partial \phi'}{\partial y}$$

$$\frac{\partial \phi'}{\partial t} = -g'H\nabla_H \cdot \vec{v}_H'$$

Here, ϕ' is:

$$\phi' = g'h'$$

A further assumption, appropriate close to the equator, is *the equatorial beta-plane approximation*, i.e., we can write *f* in Cartesian coordinates as:

$$f \approx \beta y; \quad \beta = \frac{2\Omega}{R_e}$$

Solving the linearized SWE (I)

Consider separable solutions of the form:

$$\begin{pmatrix} u' \\ v' \\ \phi' \end{pmatrix} = \begin{bmatrix} \hat{u}(y) \\ \hat{v}(y) \\ \hat{\phi}(y) \end{bmatrix} e^{i(kx-\omega t)}$$

Then:

$$-i\omega\hat{u} = \beta y\hat{v} - ik\hat{\phi}$$

$$-i\omega\hat{v} = -\beta y\hat{u} - \frac{\partial\hat{\phi}}{\partial y}$$

$$-i\omega\hat{\phi} = -g'H\left(ik\hat{u} + \frac{\partial\hat{v}}{\partial y}\right)$$

Solving the first equation above for $\hat{u}$ and substituting into the remaining two, and then eliminating $\hat{\phi}$, yields a single equation in $\hat{v}$:

$$\frac{\partial^2\hat{v}}{\partial y^2} + \left[\left(\frac{\omega^2}{g'H} - k^2 - \frac{k}{\omega}\beta\right) - \frac{\beta^2 y^2}{g'H}\right]\hat{v} = 0$$

Solving the linearized SWE (II)

$$\frac{\partial^2 \hat{v}}{\partial y^2} + \left[\left(\frac{\omega^2}{g'H} - k^2 - \frac{k}{\omega}\beta\right) - \frac{\beta^2 y^2}{g'H}\right]\hat{v} = 0$$

This equation is an example of an eigenvalue equation. Recall that an eigenvalue equation can be expressed as:

$$H(y)\psi_l(y) = \lambda_l \psi_l(y)$$

where H is an operator, is an ψ_l eigenfunction, and is an λ_l eigenvalue. The subscript *l* indicates that there may be more than 1 solution. Note that each of the eigenfunctions is orthogonal (and can be made orthonormal), i.e.,

$$\int_{-\infty}^{\infty} \psi_l^*(y)\psi_m(y) = \delta_{lm}$$

Here * denotes the "complex conjugate"

A general solution to this equation can be expressed as a linear combination of eigenfunctions:

$$\Psi = \sum_{l=1}^{\infty} a_l \psi_l(y)$$

Solving the linearized SWE (II)

For our equation, the eigenvalues are given by the following ("dispersion") relationship:

$$\frac{\sqrt{g'H}}{\beta}\left(\frac{\omega^2}{g'H} - k^2 - \frac{k}{\omega}\beta\right) = 2l + 1;\ \ l = 0,1,2,\ldots$$

l is the meridional wavenumber

The eigenfunctions are of the form:

$$\hat{v}_l(\xi) = v_0 H_l(\xi) e^{-\xi^2/2}; \quad \xi = \left(\frac{\beta}{\sqrt{g'H}}\right)^{1/2} y$$

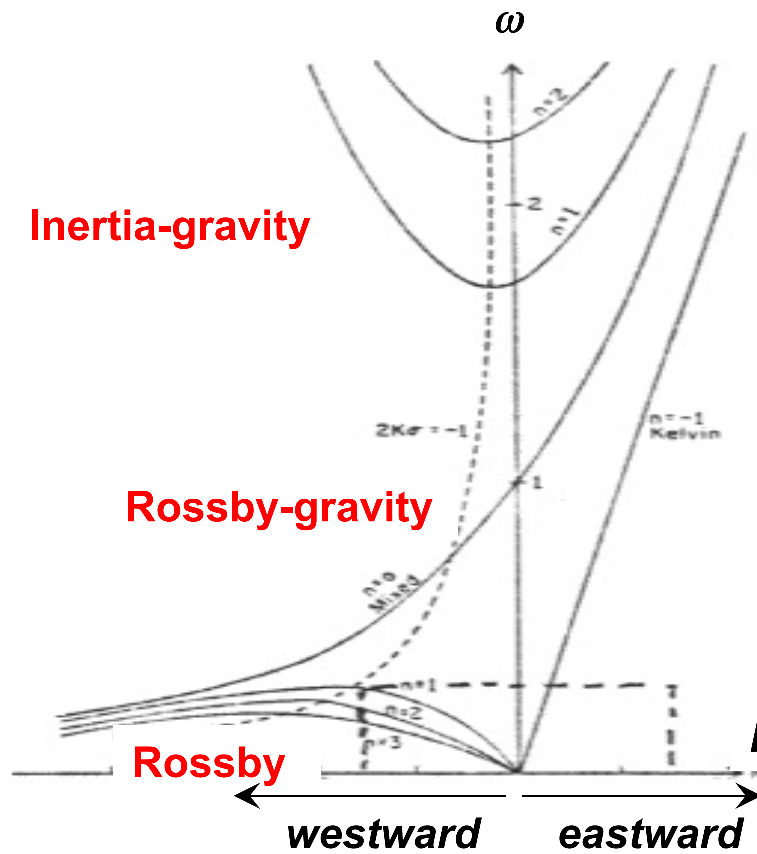
Here, the function $H_l(\xi)$ is the l^{th} Hermite polynomial. The first three Hermite polynomials are: $H_0 = 1$; $H_1 = 2\xi$; $H_2 = 4\xi^2 - 2$.

Note that the eigenfunctions decay as $|y| \rightarrow \infty$. This requirement follows from the equatorial beta-plane approximation which is not valid at high latitudes, so solutions must be "equatorially trapped".

Gravity/Rossby waves

Since the meridional dispersion relationship is cubic in ω, it has 3 solutions. These are interpreted as (i) eastward and (ii) westward propagating equatorially trapped inertia-gravity waves and (iii) westward propagating Rossby waves. The case for l=0, i.e., v is a Gaussian centered on the equator, must be treated separately from l > 0. In particular, the dispersion relationship can be written as:

$$\left(\frac{\omega}{\sqrt{g'H}} - \frac{\beta}{\omega} - k\right)\left(\frac{\omega}{\sqrt{g'H}} + k\right) = 0$$



The root coming from the second term in parentheses is not permitted, as this term is required not to vanish by the derivation. The remaining two roots correspond to an eastward propagating gravity wave and a westward propagating mode that asymptotes to Rossby wave-like behavior in the short wavenumber limit.

Equatorial Kelvin wave (I)

The equatorial Kelvin mode has no meridional velocity perturbations, so

$$-i\omega\hat{u} = -ik\hat{\phi}$$

$$0 = -\beta y\hat{u} - \frac{\partial\hat{\phi}}{\partial y}$$

$$-i\omega\hat{\phi} = -g'H(ik\hat{u})$$

Eliminating $\hat{\phi}$ between the first and third equations gives:

$$\left(\omega^2 - g'Hk^2\right)\hat{u} = 0 \Rightarrow \frac{\omega^2}{k^2} = c^2 = g'H$$

On the other hand, eliminating $\hat{\phi}$ between the first and second equations gives:

$$0 = -\beta y\hat{u} - c\frac{\partial\hat{u}}{\partial y}$$

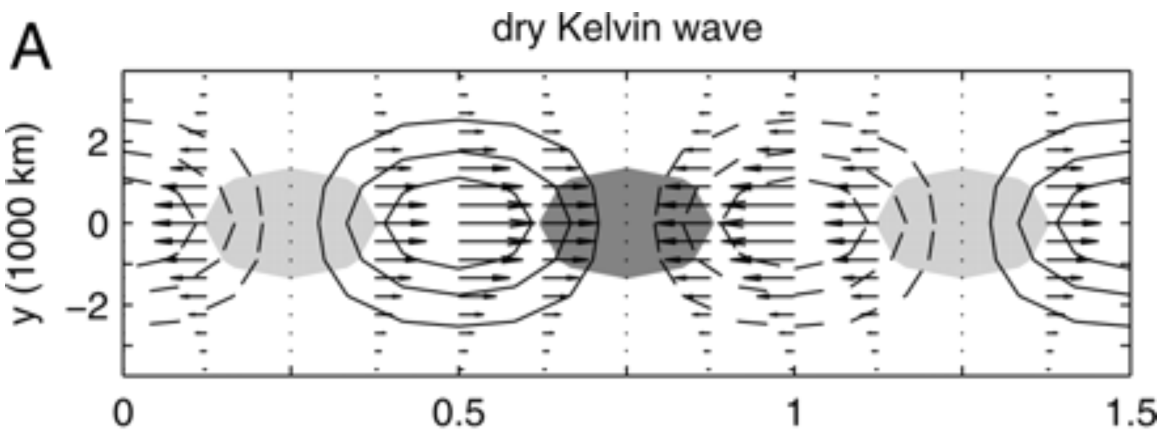
Integrating this equation gives:

$$\hat{u} = u_0 e^{(-\beta y^2/2c)}$$

Note that for this solution to decay away from the equator, *c>0*, i.e., the Kelvin wave mode propagates eastward.

Equatorial Kelvin wave (II)

The structure of the equatorial Kelvin mode looks like:



Contours show lower tropospheric pressure with positive (negative) anomalies denoted by solid (dashed) lines. The contour interval is one-fourth the maximum amplitude of the anomaly, and the zero contour is not shown. Anomalies of convergence (divergence) that are greater than two-thirds the maximum amplitude are shaded dark (light) gray. From **Majda & Stechmann 2009**.

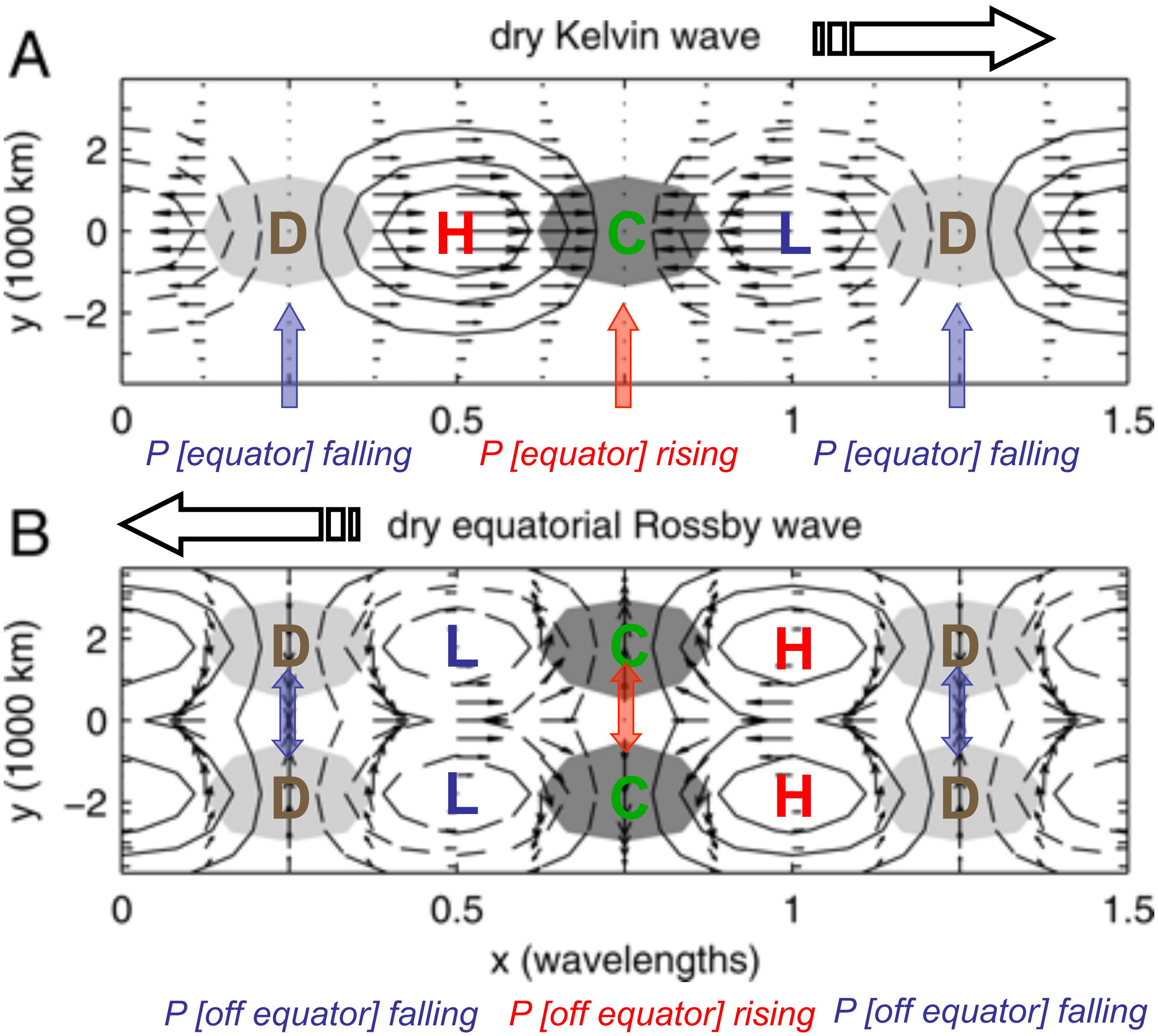
In the Kelvin wave, the meridional force balance is exact geostrophic balance between the meridional pressure gradient and zonal velocity; the equatorial Kelvin mode owes its existence to the change in sign of the Coriolis force at the equator.

The "e-folding" decay width of the Kelvin wave structure is given by:

$$Y_K = \left(\frac{2c}{\beta}\right)^{1/2}$$

For c = 30 ms^{-1}, Y_K = 1600 km.

Kelvin vs. Rossby propagation



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Rossby wave phase speeds

The (rearranged) meridional dispersion relationship for Rossby and inertia-gravity waves:

$$\frac{\omega^3}{c_0^2} - \omega\left(k^2 + \frac{(2l+1)\beta}{c_0}\right) = k\beta;\ \ l = 0,1,2,...$$

Let's examine this relationship more closely to estimate the phase speed of the associated waves. Recall that Rossby waves exist in the *low-frequency limit*, i.e., $\omega \rightarrow 0$. Thus, in the dispersion relationship, we can ignore the term in ω^3, so:

$$\frac{\omega}{k} = \frac{-\beta}{k^2 + (2l+1)(\beta / c_0)}$$

$$= -\frac{c_0}{(2l+1) + (c_0/\beta)k^2}$$

It's immediately obvious that the phase speed of Rossby waves $\frac{\omega}{k}$ is negative, or westward. For the "gravest" Rossby wave mode (l = 1), we see, in the limit of small zonal wavenumber ($k \rightarrow 0$), that the magnitude of the phase speed is $c_0/3$, or 1/3 the Kelvin wave phase speed, with higher order Rossby wave mode having smaller phase speeds. The difference in Kelvin and Rossby wave phase speeds plays an key role in the evolution of El Niño conditions.

Inertia-gravity wave phase speeds

$$\frac{\omega^3}{c_0^2} - \omega\left(k^2 + \frac{(2l+1)\beta}{c_0}\right) = k\beta;\;\; l = 0,1,2,...$$

Inertia-gravity waves exist in the *high-frequency limit*, i.e., $\omega \rightarrow \infty$. In this case, we can ignore the $k\beta$ term. Thus, for the nontrivial roots:

$$\left(\frac{\omega}{k}\right)^2 = c_0{}^2\left[1 + \frac{(2l+1)\beta}{k^2 c_0}\right]$$

$$\Rightarrow \frac{\omega}{k} = \pm c_0\left[1 + \frac{(2l+1)\beta}{k^2 c_0}\right]^{1/2}$$

Thus, the phase speed of inertia gravity waves is either positive or negative [and in this limit, equal in magnitude]. Let's briefly revisit the SWE, in 1D (say x) and in the limit of $\beta \rightarrow 0$. Then:

$$\frac{\partial u'}{\partial t} = -\frac{\partial \phi'}{\partial x} \qquad \frac{\partial \phi'}{\partial t} = -g'H\frac{\partial u'}{\partial x}$$

This is a wave equation for shallow water gravity waves with phase speed c_0, which we see immediately from the dispersion relationship with $\beta \rightarrow 0$.

Defining equivalent depth

Starting from the full 3D equations for an incompressible atmosphere with constant static stability and assuming separable solutions, we can derive a system of equations like the SWE but with vertical structure dictated by an equation of the form:

$$\frac{d^2\Psi_n}{dz^2} + \mu_n^2\Psi_n = F_n$$

Here μ_n is the vertical wavenumber, defined as:

$$\mu_n^2 = \frac{1}{H_n T_0}\left(\frac{dT_0}{dz} + \frac{g}{c_P}\right) - \frac{1}{4H_s^2}$$

$$= \frac{N^2}{gH_n} - \frac{1}{4H_s^2}$$

Here H_s is the atmospheric scale height (g/R_dT). The equivalent depth H_n is a function of wave mode. As an example, for a dry tropical wave with vertical wavelength of 5×10^3 m, with a scale height of 8×10^3 m:

$$H_n = \frac{N^2}{g[\mu_n^2 + (4H_s^2)^{-1}]} \approx \frac{N^2}{g\mu_n^2}$$

The equivalent depth is then ~10 m. The wavespeed would be $(gH_n)^{1/2}$=10 ms^{-1}

Summary of tropical wave speeds and spatial scales

	Dry wave speed (ms^{-1})	Moist* wave speed (ms^{-1})	Spatial scale (km)
Kelvin	30-60	12-25	5-10 x 10^3 (zonal); 2x10^3 (meridional)
Rossby	10-20	5-7	4-10 x 10^3 (zonal); 1x10^3 (meridional)
Rossby-gravity	15-20	8-10	1-4 x 10^3 (zonal)

*Moist wave speeds, associated with coupling to moist convection, reduces phase speed through latent heat release ahead of the wave, making the environment effectively more buoyant [i.e., lowering N].

Wheeler-Kiladis diagram